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LETTER TO THE EDITOR

A new type of solution of the Schrödinger equation on a self-similar fractal potential

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Abstract
Scattering of a quantum particle by a self-similar fractal potential on a Cantor set is investigated. We present a new type of solution of the functional equation for the transfer matrix of this potential, which was derived earlier from the Schrödinger equation.

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In this letter, we address the model [1, 2] of scattering a quantum particle by a self-similar fractal potential (SSFP) given on a Cantor set. This scattering problem is, perhaps, the most simple one to allow studying the influence of the scale invariance of ideal deterministic fractals on physical processes in continuous media to involve such fractal structures.

Note that the sharp attenuations, found in [1], in the spectrum of probability waves transmitted through this ideal fractal potential have also been observed experimentally (see [3]) in the transmission spectrum of electromagnetic waves propagating through a real fractal medium (see a numerical modelling for the corresponding pre-fractals in [4]). However, the problem is that the model [1, 2] remains incomplete in some respects. In this letter, we present a new type of solution to the Schrödinger equation on the SSFP, in addition to two types presented in [1, 2].

So, let $V(x)$ be an SSFP on the generalized Cantor set in the interval $[0, L]$; each level of the SSFP consists of N ($N \geqslant 2$) SSFPs of the next level whose width is α times smaller than that of the former (see [2]). Let W be the power of the SSFP, that is, its total area: $W = \int_{-\infty}^{\infty} V(x)\,dx$. In line with [1, 2], for a particle with a given energy E ($E = \hbar^2k^2/2m$), the transfer matrix $Z(\phi)$ ($\phi = kL$) of the SSFP must obey the functional equation

$$\mathbf{Z}(\phi) = \mathbf{Z}(\alpha\phi)\,[\mathbf{D}(\gamma\phi)\mathbf{Z}(\alpha\phi)]^{N-1},$$

$$\mathbf{Z}(\phi) = \begin{pmatrix} q(\phi) & p(\phi) \\ p^*(\phi) & q^*(\phi) \end{pmatrix}, \qquad \mathbf{D}(\phi) = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix}, \tag{1}$$

$$q(\phi) = \frac{1}{\sqrt{T(\phi)}} \exp[-iJ(\phi)], \qquad p(\phi) = \sqrt{\frac{R(\phi)}{T(\phi)}} \exp\left[i\left(\frac{\pi}{2} + F(\phi)\right)\right].$$

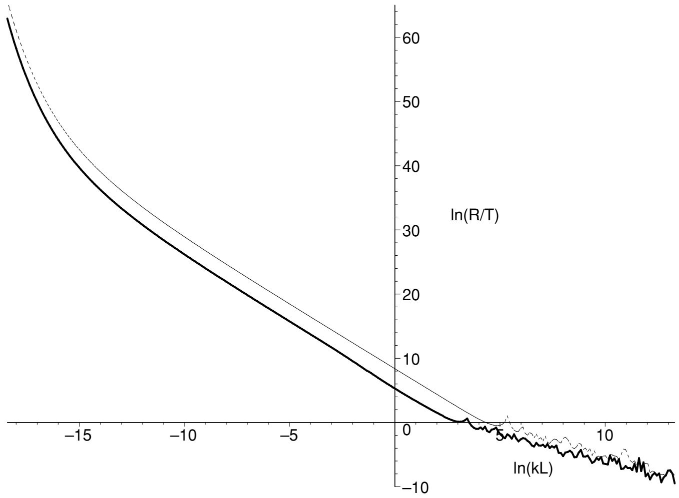

Figure 1. The $\ln(\phi)$ dependence of $\ln(R/T)$ for $s = 0.5$, $c = 0.001$ and $\omega = 10$; bold full curve: $N = 2$; thin full curve: $N = 4$.

Here $\gamma = \frac{\alpha - N}{\alpha(N-1)}$, $R = 1 - T$, and $T(\phi)$, $J(\phi)$ and $F(\phi)$ are, respectively, the transmission coefficient and phase characteristics of the SSFP; $F = 0$ for the SSFP barriers and $F = \pi$ for the SSFP wells (see [2]).

As it has turned out, equation (1) does not uniquely determine the transfer matrix of the SSFP. Two different types of solutions of this equation have been presented in [1, 2]. Recall that the first type was obtained for any values of W, α and N to characterize the SSFP. In this case, for small values of ϕ, $\sqrt{T(\phi)} \sim y(\phi) \sim \phi^s$, where s is the fractal dimension of the Cantor set: $s = \ln(N)/\ln(\alpha)$, $y = \frac{\pi}{2} - J$. The second type of solutions exists only for the SSFP barriers, if $W = \frac{3N\hbar^2}{mL}$. For this type, $\sqrt{T(\phi)} \sim y(\phi) \sim \phi$ for small values of ϕ.

In this letter, we present a new (third) type of solutions (found by Chuprikov), with a cardinally different behaviour of the tunnelling parameters in the asymptotic region. Namely, in this case, for small values of ϕ we have

$$T(\phi) = \{1 + \cosh^2[\omega(\ln(\phi))]\sinh^2(c\phi^{-s})\}^{-1},$$
$$J(\phi) = \arctan\{\sinh[\omega(\ln(\phi))]\tanh(c\phi^{-s})\}, \tag{2}$$

where c is a nonzero constant, and ω is a nonzero real-valued function to obey the condition $\omega[\ln(\phi)] = \omega[\ln(\phi) + \ln(\alpha)]$.

To extend this solution onto the whole $\ln(\phi)$-axis, one has to use the recurrence relations (18) and (19) presented in [2]. As in [1, 2], we display here $\ln(R/T)$ versus $\ln(\phi)$ (see figures 1 and 2).

In the numerical study we took the function $\omega(\phi)$ to be constant. Our calculations showed that in this case varying the parameter ω, in a wide region, does not practically influence the tunnelling parameters of the SSFP. A simple analysis shows that the tunnelling parameters are non-differentiable functions at point $\phi = 0$, when ω depends on ϕ.

As is seen from figures 1 and 2, there are three regions on the $\ln(\phi)$-axis, with a qualitatively different dependence of $\ln(R/T)$:

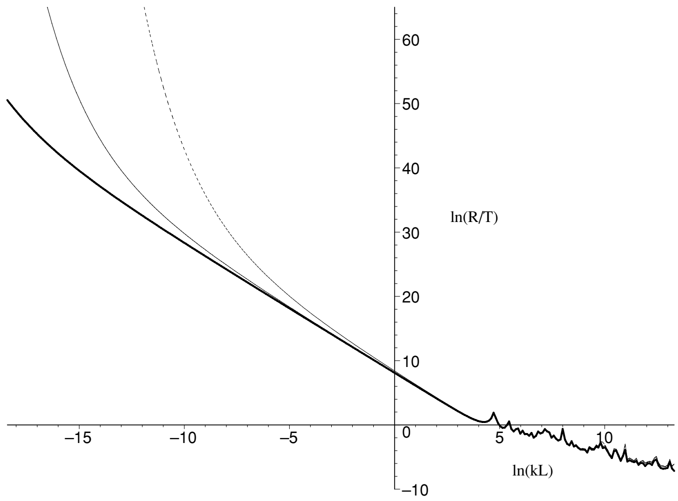

Figure 2. The $\ln(\phi)$ dependence of $\ln(R/T)$ for $N = 3, \alpha = 13$ and $\omega = 10$; broken curve: $c = 0.1$; thin full curve: $c = 0.01$; bold full curve: $c = 0.001$.

- In the left region

$$\ln\ln\left(\frac{R}{T}\right) \sim \ln(2|c|) - s\ln(\phi).$$

- In the middle region

$$\ln\left(\frac{R}{T}\right) \sim -2\ln(\phi).$$

- In the right region

$$\ln(\widetilde{R/T}) \sim -2s\ln(\phi),$$

where $\widetilde{R/T}$ is the envelop of $R(\phi)/T(\phi)$.

Note that the right region appears for all three types of solutions, but the middle region appears only for the third and second types (see [2]). As regards the left one to follow from (2), such a behaviour is a distinctive feature of the third type of solutions.

It is also important to note here that, for the solutions of the first and third types, the phase path of the wave inside the out-of-barrier regions (i.e., in the regions where the potentials are equal to zero) is infinitesimally small in comparison with the wave path in the barrier regions. This feature distinguishes these types of solutions from the second one.

So, there are at least three types of the transfer matrices of the SSFP. As is seen, though all of them are nonzero only on the Cantor set, i.e., the set of zero measure, we deal with different potentials. The Cantor set is a non-countable one, and thus it yet provides a much enough room for setting potentials with such different scattering properties.

Of course, in this case, it is of great importance to find the sequences of pre-fractals to lead to the SSFPs, when the generation number of pre-fractals tends to infinity. Additionally, another open question regarding the model is that the parameters to enter the third type of solutions remain to be connected to the SSFP parameters.

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